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Weak and strong k-connectivity games

Asaf Ferber [a], Dan Hefetz [b]

[a] School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, 69978, Israel

[b] School of Mathematics, University of Birmingham, Edgbaston, Birmingham B15 2TT, United Kingdom

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ABSTRACT

For a positive integer k, we consider the k-vertex-connectivity game, played on the edge set of K_n, the complete graph on n vertices. We first study the Maker–Breaker version of this game and prove that, for any integer $k \geq 2$ and sufficiently large n, Maker has a strategy to win this game within $\lfloor kn/2 \rfloor + 1$ moves, which is easily seen to be best possible. This answers a question from Hefetz et al. (2009) [6]. We then consider the strong k-vertex-connectivity game. For every positive integer k and sufficiently large n, we describe an explicit first player's winning strategy for this game.

1. Introduction

Let X be a finite set, and let $\mathcal{F} \subseteq 2^X$ be a family of subsets. In the *strong game* $(X, \mathcal{F})$, two players, called Red and Blue, take turns in claiming one previously unclaimed element of X, with Red going first. The winner of the game is the first player to fully claim some $F \in \mathcal{F}$. If neither player is able to fully claim some $F \in \mathcal{F}$ by the time every element of X has been claimed by some player, the game ends in a *draw*. The set X will be referred to as the *board* of the game and the elements of $\mathcal{F}$ will be referred to as the *winning sets*.

It is well known from classic Game Theory that, for every strong game $(X, \mathcal{F})$, either Red has a winning strategy (that is, is able to win the game against any strategy of Blue) or Blue has a drawing strategy (that is, is able to avoid losing the game against any strategy of Red; a *strategy stealing* argument shows that Blue cannot win the game). For certain games, a hypergraph coloring argument can be used to prove that a draw is impossible and thus these games are won by Red. However, the aforementioned arguments are purely existential. That is, even if it is known that Red has a winning strategy for some strong game $(X, \mathcal{F})$, it might be very hard to describe such a strategy explicitly. The

E-mail addresses: ferberas@post.tau.ac.il (A. Ferber), d.hefetz@bham.ac.uk (D. Hefetz).

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few examples of natural games for which an explicit winning strategy is known include the *perfect matching* and *Hamilton cycle* games (see [3]).

Partly due to the great difficulty of studying strong games, weak games were introduced. In the *Maker–Breaker game* (also known as *weak* game) $(X, \mathcal{F})$, two players, called Maker and Breaker, take turns in claiming previously unclaimed elements of X, with Breaker going first. Each player claims *exactly* one element of X per turn. The set X is called the *board* of the game and the members of $\mathcal{F}$ are referred to as the *winning sets*. Maker wins the game as soon as he occupies all elements of some winning set. If Maker does not fully occupy any winning set by the time every board element is claimed by some player, then Breaker wins the game. Note that being the first player is never a disadvantage in a Maker–Breaker game (see e.g. [2]). Hence, in order to prove that Maker can win some Maker–Breaker game as the first or second player, it suffices to prove that he can win this game as the second player.

In this paper, we study the weak and strong versions of the k-vertex-connectivity game $(E(K_n), \mathcal{C}_n^k)$. The board of this game is the edge set of the complete graph on n vertices, and its family of winning sets $\mathcal{C}_n^k$ consists of the edge sets of all k-vertex-connected spanning subgraphs of K_n.

It is easy to see (and it also follows from [8]) that, for every $n \geq 4$, Maker can win the weak game $(E(K_n), \mathcal{C}_n^1)$ within $n-1$ moves. Clearly this is best possible. It follows from [7] that, if n is not too small, then Maker can win the weak game $(E(K_n), \mathcal{C}_n^2)$ within $n+1$ moves, and that this is best possible as well. It was proved in [6] that, for every fixed $k \geq 3$ and sufficiently large n, Maker can win the weak game $(E(K_n), \mathcal{C}_n^k)$ within $kn/2 + (k+4)(\sqrt{n} + 2n^{2/3} \log n)$ moves. Since clearly Maker cannot win this game in less than $kn/2$ moves, this shows that the number of excess moves Maker plays is $o(n)$. It was asked in [6] whether the dependence on n of the number of excess moves can be omitted, that is, whether Maker can win $(E(K_n), \mathcal{C}_n^k)$ within $kn/2 + c_k$ moves for some c_k which is independent of n. We answer this question in the affirmative.

Theorem 1.1. *Let $k \geq 2$ be an integer, and let n be a sufficiently large integer. Then Maker has a strategy to win the weak game $(E(K_n), \mathcal{C}_n^k)$ within at most $\lfloor kn/2 \rfloor + 1$ moves.*

In the minimum-degree-k game $(E(K_n), \mathcal{D}_n^k)$, the board is again the edge set of K_n, and the family of winning sets $\mathcal{D}_n^k$ consists of the edge sets of all subgraphs of K_n with minimum degree at least k. Since $\mathcal{C}_n^k \subseteq \mathcal{D}_n^k$ for every k and n, we immediately obtain the following result.

Corollary 1.2. *Let $k \geq 1$ be an integer, and let n be a sufficiently large integer. Then Maker has a strategy to win the weak game $(E(K_n), \mathcal{D}_n^k)$ within at most $\lfloor kn/2 \rfloor + 1$ moves.*

Note that, for $k = 1$, Corollary 1.2 does not follow from Theorem 1.1. However, this case was proved in [6]. Moreover, we will prove a strengthening of this result in Section 3.

Note that both Theorem 1.1 and Corollary 1.2 are best possible. Indeed, assume for the sake of contradiction that Maker has a strategy to build a subgraph of K_n with minimum degree at least k within at most $\lfloor kn/2 \rfloor$ moves. It follows that kn is even since, if kn is odd, then every graph on n vertices and at most $\lfloor kn/2 \rfloor$ edges has a vertex of degree at most $k-1$. Since Maker wins in $kn/2$ moves, he must do so by building a k-regular spanning subgraph of K_n. Let G denote the graph he builds in the first $kn/2 - 1$ moves. Then there are two vertices $x, y \in V(K_n)$ of degree $k-1$ in G, every other vertex of K_n has degree k in G, and $xy \notin E(G)$. In his $(kn/2)$th move, Breaker claims xy, and thus Maker cannot enlarge the degree of both x and y in one additional move, contrary to our assumption.

It was observed in [3] that a fast winning strategy for Maker in the weak game $(X, \mathcal{F})$ has the potential of being used to devise a winning strategy for the first player in the strong game $(X, \mathcal{F})$. Using our strategy for the weak game $(E(K_n), \mathcal{C}_n^k)$, we will devise an explicit winning strategy for the corresponding strong game. We restrict our attention to the case $k \geq 3$, as the (much simpler) cases $k = 1$ and $k = 2$ were discussed in [3].

Theorem 1.3. *Let $k \geq 3$ be an integer, and let n be a sufficiently large integer. Then Red has a strategy to win the strong game $(E(K_n), \mathcal{C}_n^k)$ within at most $\lfloor kn/2 \rfloor + 1$ moves.*

Our proof of Theorem 1.3 will in fact show that Red can build a k-vertex-connected graph before Blue can build a graph with minimum degree at least k. We thus have the following corollary.

Corollary 1.4. *Let $k \geq 1$ be an integer, and let n be a sufficiently large integer. Then Red has a strategy to win the strong game $(E(K_n), \mathcal{D}_n^k)$ within at most $\lfloor kn/2 \rfloor + 1$ moves.*

As with Corollary 1.2, the cases $k = 1$ and $k = 2$ do not follow from Theorem 1.3. However, these simple cases were discussed in [3]. Moreover, for $k = 1$, we will prove a strengthening of this result in Section 3.

The rest of this paper is organized as follows. In Section 1.1, we introduce some notation and terminology that will be used throughout this paper. In Section 2, we describe a family of k-vertex-connected graphs that will be used in the proofs of Theorems 1.1 and 1.3. In Section 3, we study certain simple games; the results obtained will be used in the following sections. In Section 4, we prove Theorem 1.1, and in Section 5 we prove Theorem 1.3. Finally, in Section 6, we present some open problems.

1.1. Notation and terminology

Our graph-theoretic notation is standard and follows that of [9]. In particular, we use the following.

For a graph G, let $V(G)$ and $E(G)$ denote its sets of vertices and edges, respectively, and let $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For disjoint sets $A, B \subseteq V(G)$, let $E_G(A, B)$ denote the set of edges of G with one endpoint in A and one endpoint in B, and let $e_G(A, B) = |E_G(A, B)|$. For a set $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G which is induced on the set S. For disjoint sets $S, T \subseteq V(G)$, let $N_G(S, T) = \{u \in T : \exists v \in S, uv \in E(G)\}$ denote the set of neighbors of the vertices of S in T. For a set $T \subseteq V(G)$ and a vertex $w \in V(G)$, we abbreviate $N_G(\{w\}, T \setminus \{w\})$ to $N_G(w, T)$, and let $d_G(w, T) = |N_G(w, T)|$ denote the degree of w into T. For a set $S \subseteq V(G)$ and a vertex $w \in V(G)$, we abbreviate $N_G(S, V(G) \setminus S)$ to $N_G(S)$ and $N_G(w, V(G) \setminus \{w\})$ to $N_G(w)$. We let $d_G(w) = |N_G(w)|$ denote the degree of w in G. The minimum and maximum degrees of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For vertices $u, v \in V(G)$, let $dist_G(u, v)$ denote the *distance* between u and v in G, that is, the number of edges in a shortest path of G, connecting u and v. Often, when there is no risk of confusion, we omit the subscript G from the notation above. For a positive integer k, let $[k]$ denote the set $\{1, \ldots, k\}$.

Assume that some Maker–Breaker game, played on the edge set of some graph G, is in progress. At any given moment during this game, we denote the graph spanned by Maker's edges by M and the graph spanned by Breaker's edges by B. At any point during the game, the edges of $G \setminus (M \cup B)$ are called *free*.

Similarly, assume that some strong game, played on the edge set of some graph G, is in progress. At any given moment during this game, we denote the graph spanned by Red's edges by R and the graph spanned by Blue's edges by B. At any point during the game, the edges of $G \setminus (R \cup B)$ are called *free*.

2. A family of k-vertex-connected graphs

In this section, we describe a family of k-vertex-connected graphs. We will use this family in the proofs of Theorems 1.1 and 1.3.

Let $k \geq 3$ be an integer, and let n be a sufficiently large integer. Let $\mathcal{G}_k$ be the family of all graphs $G_k = (V, E_k)$ on n vertices for which there exists a partition $V = V_1 \cup \cdots \cup V_{k-1}$ such that all of the following properties hold.

(i) $|V_i| \geq 5$ for every $1 \leq i \leq k - 1$.
(ii) $\delta(G_k) \geq k$.
(iii) $G_k[V_i]$ admits a Hamilton cycle C_i for every $1 \leq i \leq k - 1$.
(iv) For every $1 \leq i < j \leq k - 1$, the bipartite subgraph of G_k with parts V_i and V_j admits a matching of size 3.
(v) $|\{j \in [k-1] \setminus \{i\} : d_{G_k}(u, V_j) = 0\}| \leq 1$ holds for every $1 \leq i \leq k - 1$ and every $u \in V_i$.
(vi) For every $1 \leq i \leq k - 1$ and every $u, v \in V_i$, if $|\{j \in [k-1] \setminus \{i\} : d_{G_k}(u, V_j) = 0\}| = |\{j \in [k-1] \setminus \{i\} : d_{G_k}(v, V_j) = 0\}| = 1$, then $dist_{C_i}(u, v) \geq 2$.

Proposition 2.1. *For every integer $k \geq 3$ and sufficiently large integer n, every $G_k \in \mathcal{G}_k$ is k-vertex-connected.*

Proof. Let G_k be any graph in $\mathcal{G}_k$. Let $S \subseteq V$ be an arbitrary set of size $k-1$. We will prove that $G_k \setminus S$ is connected. We distinguish between the following three cases.

Case1: $|S \cap V_i| = 1$ for every $1 \leq i \leq k-1$.

Since $G_k[V_i]$ is Hamiltonian for every $1 \leq i \leq k-1$ by Property (iii) above, it follows that $(G_k \setminus S)[V_i]$ is connected for every $1 \leq i \leq k-1$. Hence, in order to prove that $G_k \setminus S$ is connected, it suffices to prove that $E_{G_k \setminus S}(V_i, V_j) \neq \emptyset$ holds for every $1 \leq i < j \leq k-1$. Fix some $1 \leq i < j \leq k-1$. It follows by Property (iv) above that there exist vertices $x_i, y_i, z_i \in V_i$ and $x_j, y_j, z_j \in V_j$ such that $x_i x_j, y_i y_j, z_i z_j \in E_{G_k}(V_i, V_j)$. Clearly, at least one of these edges is present in $G_k \setminus S$.

Case2: There exist $1 \leq i < j \leq k-1$ such that $S \cap V_i = \emptyset$ and $S \cap V_j = \emptyset$.

It follows by Properties (iii) and (iv) above that $(G_k \setminus S)[V_i \cup V_j]$ is connected. Moreover, it follows by Property (v) above that $V_i \cup V_j$ is a dominating set of G_k. Hence, $G_k \setminus S$ is connected in this case.

Case3: There exist $1 \leq i \neq j \leq k-1$ such that $S \cap V_i = \emptyset$, $|S \cap V_j| = 2$, and $|S \cap V_t| = 1$ for every $t \in [k-1] \setminus \{i, j\}$.

It follows by Property (iii) above that $(G_k \setminus S)[V_i]$ is connected. Hence, in order to prove that $G_k \setminus S$ is connected, it suffices to prove that, for every vertex $u \in V \setminus (V_i \cup S)$, there is a path in $G_k \setminus S$ between u and some vertex of V_i. Assume first that $u \in V_t$ for some $t \in [k-1] \setminus \{i, j\}$. As in Case 1, $(G_k \setminus S)[V_t]$ is connected and $E_{G_k \setminus S}(V_t, V_i) \neq \emptyset$. It follows that the required path exists. Assume then that $u \in V_j$. If $d_{G_k}(u, V_i) > 0$, then there is nothing to prove, since $S \cap V_i = \emptyset$. Assume then that $d_{G_k}(u, V_i) = 0$; it follows by Property (v) above that $d_{G_k}(u, V_t) > 0$ holds for every $t \in [k-1] \setminus \{i, j\}$. If $d_{G_k \setminus S}(u, V_t) > 0$ holds for some $t \in [k-1] \setminus \{i, j\}$, then the required path exists as $(G_k \setminus S)[V_t]$ is connected and, as previously noted, there is an edge of $G_k \setminus S$ between V_t and V_i. Assume then that $d_{G_k \setminus S}(u, V_t) = 0$ holds for every $t \in [k-1] \setminus \{i, j\}$. It follows by Property (ii) above that $d_{G_k}(u, V_j) \geq 3$, and thus $d_{G_k \setminus S}(u, V_j) \geq 1$. Let $w \in V_j \setminus S$ be a vertex such that $uw \in E_k$. If $d_{G_k}(w, V_i) > 0$, then the required path exists. Otherwise, since $|V_j| \geq 5$ by Property (i) above, it follows by Property (vi) above that there exists a vertex $z \in N_{G_k \setminus S}(u, V_j) \cup N_{G_k \setminus S}(w, V_j)$ such that $d_{G_k}(z, V_i) > 0$. Hence, the required path exists.

We conclude that G_k is k-vertex-connected. □

Note that, while $\mathcal{G}_k$ includes very dense graphs, such as K_n, for every $k \geq 3$ and every sufficiently large n, this family also includes graphs with $\lceil kn/2 \rceil$ edges; that is, k-vertex-connected graphs which are as sparse as possible. One illustrative example of such a graph consists of $k-1$ pairwise vertex disjoint cycles, each of length $n/(k-1)$, where every pair of cycles is connected by a perfect matching (in particular, $k-1 \mid n$). The graphs Maker and Red will build in the proofs of Theorems 1.1 and 1.3, respectively, are fairly similar to this example.

3. Auxiliary games

In this section, we consider several simple games. Some might be interesting in their own right, whereas others are quite artificial. The results we prove about these games will be used in our proofs of Theorems 1.1 and 1.3. We divide this section into several subsections, each discussing one game.

3.1. A large matching game

In this subsection, we study a game whose family of winning sets is not monotone increasing and depends on the elements claimed by both players. It is thus not a weak (or strong) game as defined in the introduction. Nevertheless, by abuse of terminology, we refer to it as a weak game and to the players as Maker and Breaker since it will be used by Maker in the proof of Theorem 1.1.

Let $G = (V_1 \cup V_2, E)$ be a bipartite graph, let $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$, and let d be a positive integer. The board of the weak game $G(V_1, U_1; V_2, U_2; d)$ is E. Maker wins this game if and only if he accomplishes all of the following goals.

(i) Maker's graph is a matching.

(ii) $d_M(u) = 1$ for every $u \in (V_1 \setminus U_1) \cup (V_2 \setminus U_2)$.
(iii) $d_M(u) = 1$ for every $u \in V_1 \cup V_2$ for which $d_B(u) \geq d$.
(iv) $|\{u \in U_1 : d_M(u) = 0\}| \geq |U_1|/2$ and $|\{u \in U_2 : d_M(u) = 0\}| \geq |U_2|/2$.

Lemma 3.1. *Let m be a non-negative integer, let d be a positive integer, let $d^{-1} \leq \varepsilon \leq 0.1$ be a real number, and let $n_0 = n_0(m, d, \varepsilon)$ be a sufficiently large integer. Let $G = (V_1 \cup V_2, E)$ be a bipartite graph which satisfies all of the following properties.*

(P1) $n_0 \leq |V_1| \leq |V_2| \leq (1+\varepsilon)|V_1|$.
(P2) $d_G(u, V_2) \geq |V_2| - m$ *for every* $u \in V_1$.
(P3) $d_G(u, V_1) \geq |V_1| - m$ *for every* $u \in V_2$.

Let $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$ be such that $10\varepsilon|V_1| \leq |U_1| \leq 11\varepsilon|V_1|$ and $10\varepsilon|V_2| \leq |U_2| \leq 11\varepsilon|V_2|$. Then Maker has a winning strategy for the game $G(V_1, U_1; V_2, U_2; d)$.

Proof. First, we describe a strategy for Maker, and then we prove that it is a winning strategy. At any point during the game, if Maker is unable to follow the proposed strategy, then he forfeits the game.

Throughout the game, Maker maintains a matching M_G and a set $D \subseteq V_1 \cup V_2$ of *dangerous* vertices, where a vertex $v \in V_1 \cup V_2$ is called dangerous if $d_M(v) = 0$ and $d_B(v) \geq d$. Initially, $M_G = D = \emptyset$.

For every positive integer j, Maker plays his jth move as follows.

(1) If $D \neq \emptyset$, then Maker claims an arbitrary free edge $uv \in E$ for which $u \in D$ and $d_M(v) = 0$. Subsequently, he updates $M_G := M_G \cup \{uv\}$ and $D := D \setminus \{u, v\}$.
(2) Otherwise, if there exists a free edge $uv \in E$ such that $u \in V_1 \setminus U_1$, $v \in V_2 \setminus U_2$ and $d_M(u) = d_M(v) = 0$, then Maker claims it. Subsequently, he updates $M_G := M_G \cup \{uv\}$.
(3) Otherwise, if there exists a vertex $u \in (V_1 \setminus U_1) \cup (V_2 \setminus U_2)$ such that $d_M(u) = 0$, then Maker claims a free edge $uv \in E$ such that $d_M(v) = 0$. Subsequently, he updates $M_G := M_G \cup \{uv\}$.

The game is over as soon as M_G covers $(V_1 \setminus U_1) \cup (V_2 \setminus U_2)$ and $D = \emptyset$.

It remains to prove that Maker can indeed follow the proposed strategy without forfeiting the game and that, by doing so, he wins the game $G(V_1, U_1; V_2, U_2; d)$.

We begin by showing that, even if he forfeits the game, Maker accomplishes goals (i) and (iv).

Claim 3.2. *Goals* (i) *and* (iv) *are met at any point during the game.*

Proof. It readily follows from the description of the proposed strategy that Maker's graph is a matching at any point during the game. Hence, he accomplishes goal (i). Next, we prove that Maker accomplishes goal (iv) as well. Maker does not match any vertex of $U_1 \cup U_2$ in Part (2). During Parts (1) and (3), Maker matches at most one vertex of U_1 and at most one vertex of U_2 per move. It thus suffices to bound the number of times he plays according to the proposed strategy for these parts. Note first that, since goal (i) is met at any point during the game, the entire game lasts at most $|V_1|$ moves. In particular, Breaker can create at most $2|V_1|/d \leq 2\varepsilon|V_1| = 2\varepsilon \min\{|V_1|, |V_2|\} \leq \min\{|U_1|/5, |U_2|/5\}$ dangerous vertices throughout the game. Since Maker decreases the size of D whenever he follows Part (1) of the proposed strategy, we conclude that he follows this part at most $\min\{|U_1|/5, |U_2|/5\}$ times. Whenever Maker follows Part (3) of the proposed strategy, $D = \emptyset$, and there is no free edge $uv \in E$ such that $u \in V_1 \setminus U_1$, $v \in V_2 \setminus U_2$ and $d_M(u) = d_M(v) = 0$. It follows by these conditions and by Properties (P2) and (P3) that M_G covers at least $|V_1 \setminus U_1| - |U_1|/10$ of the vertices of $V_1 \setminus U_1$. Indeed, assume for the sake of contradiction that there exists a set $A \subseteq V_1 \setminus U_1$ such that $|A| \geq |U_1|/10$ and $d_M(u) = 0$ holds for every $u \in A$. If there exists a vertex $v \in V_2 \setminus U_2$ such that $d_M(v) = 0$, then, since $D = \emptyset$ and $|U_1|/10 \geq m+d$, it follows by Property (P3) that there exists some $u \in A$ such that uv is free. Maker should thus follow Part (2) of the proposed strategy, contrary to our assumption that he follows Part (3). Assume then that $d_M(v) = 1$ holds for every $v \in V_2 \setminus U_2$. It follows that $|M_G| \geq |V_2 \setminus U_2| \geq (1 - 11\varepsilon)|V_2| \geq (1 - 11\varepsilon)|V_1| \geq |V_1 \setminus U_1| - \varepsilon|V_1| \geq |V_1 \setminus U_1| - |U_1|/10$, as claimed. It follows that, while playing according to the proposed strategy for Part (3), Maker matches at most $|U_1|/10 \leq |U_2|/5$ vertices of U_2, where the inequality holds by Property (P1) and the assumed bounds on $|U_1|$ and $|U_2|$. A similar argument (whose details we omit) shows that, while playing according to the strategy for Part (3), Maker matches at most $|U_2|/5 \leq 3|U_1|/10$ vertices of U_1. We conclude that throughout the game Maker matches at most $|U_1|/2$ vertices of U_1 and at most $|U_2|/2$ vertices of U_2. $\square$

It readily follows from its description that Maker can play according to Part (2) of the proposed strategy. Moreover, since by Claim 3.2 he accomplishes goal (iv), it follows that he can play according to Part (3) of the proposed strategy as well. Finally, since he accomplishes goal (iv) and since he follows Part (1) at most $\min\{|U_1|/5, |U_2|/5\}$ times, it follows that he can play according to Part (1) of the proposed strategy. We conclude that Maker can follow the proposed strategy, and thus accomplishes goals (ii) and (iii) as well. $\square$

3.2. A weak positive minimum degree game

In this subsection, we study the weak *positive minimum degree* game $(E(G), \mathcal{D}_G^1)$, played on the edge set of some given graph G. The family of winning sets $\mathcal{D}_G^1$ consists of the edge sets of all spanning subgraphs of G with minimum degree at least 1. Recall that in the special case $G = K_n$ we denoted this family by $\mathcal{D}_n^1$. The following result was proved in [6].

Theorem 3.3 (*[6, Corollary 1.3]*). *For sufficiently large n, Maker has a strategy to win the weak game $(E(K_n), \mathcal{D}_n^1)$ within $\lfloor n/2 \rfloor + 1$ moves.*

We strengthen Theorem 3.3 by proving that its assertion holds even when the board is not complete, though still very dense.

Theorem 3.4. *For every positive integer m there exists an integer n_m such that, for every $n \geq n_m$ and for every graph $G = (V, E)$ on n vertices with minimum degree at least $n - m$, Maker (as the first or second player) has a strategy to win the weak positive minimum degree game $(E(G), \mathcal{D}_G^1)$, within at most $\lfloor n/2 \rfloor + 1$ moves.*

Proof. We prove Theorem 3.4 by induction on m. At any point during the game, let $V_0 := \{u \in V : d_M(u) = 0\}$ denote the set of vertices of G which are isolated in Maker's graph, and let $H := (B \cup (K_n \setminus G))[V_0]$.

In the induction step, we will need to assume that $m \geq 3$. Hence, we first consider the cases $m = 1$ and $m = 2$ separately. If $m = 1$, then $G = K_n$, and thus the result follows immediately by Theorem 3.3. Assume then that $m = 2$, and assume for convenience that n is even (the proof for odd n is similar, and in fact slightly simpler; we omit the straightforward details). For every $1 \leq i \leq n/2 - 1$, in his ith move, Maker claims a free edge uv such that $u, v \in V_0$ and $d_H(u) = \Delta(H)$. In each of his next two moves, Maker claims a free edge xy such that $x \in V_0$ and $y \in V$.

It is evident that, if Maker is able to follow this strategy, then he wins the positive minimum degree game $(E(G), \mathcal{D}_G^1)$, within $\lfloor n/2 \rfloor + 1$ moves. It thus remains to prove that he can indeed do so. In order to show that he can follow the first $n/2 - 1$ moves of the proposed strategy, we first prove by induction on n that $\Delta(H) \leq 1$ holds immediately before Breaker's ith move for every $1 \leq i \leq n/2 - 1$. This holds for $i = 1$ by assumption. Assume that it holds for some positive integer i. Clearly $\Delta(H) \leq 2$ holds immediately after Breaker's ith move. Moreover, there are at most two vertices of V_0 whose degree in H is 2, and if there are exactly two such vertices, then they are connected by an edge of Breaker. In his ith move, Maker claims an edge which is incident with a vertex of maximum degree in H. It follows that $\Delta(H) \leq 1$ holds immediately after Maker's ith move.

Since, for every $1 \leq i \leq n/2-1$, immediately before Maker's ith move we have $|V_0| = n-2(i-1) \geq 4$ and $\Delta(H) \leq 2$, Maker can play his ith move according to the proposed strategy. Moreover, it is clear that Maker can play his $(n/2)$th and $(n/2 + 1)$th (unless he already wins after $n/2$ moves) moves according to the proposed strategy.

Assume then that $m \geq 3$ and that the assertion of the theorem holds for $m - 1$. We present a fast winning strategy for Maker. At any point during the game, if Maker is unable to follow the proposed strategy, then he forfeits the game. The strategy is divided into the following two stages.

Stage I: Maker builds a matching while trying to decrease $\Delta(H)$. In every move, Maker claims a free edge uv such that $u, v \in V_0$, $d_H(u) = \Delta(H)$ and $d_H(v) = \max\{d_H(w) : w \in V_0 \text{ and } uw \in E(G \setminus B)\}$. This stage is over as soon as $\Delta(H) \leq m - 2$ first holds.

Stage II: Maker builds a spanning subgraph of $G[V_0]$ with positive minimum degree within $\lfloor |V_0|/2 \rfloor + 1$ moves.

It is evident that, if Maker can follow the proposed strategy without forfeiting the game, then he wins the positive minimum degree game on G within $\lfloor n/2 \rfloor + 1$ moves. It thus suffices to prove that he can indeed do so. First we prove that Maker can follow Stage I of his strategy, and, moreover, that this stage lasts at most $\frac{(m-1)n}{2m} + 2$ moves. It is clear from the description of Maker's strategy that the following property is maintained throughout Stage I.

($*$) $\Delta(H) \leq m$ holds after every move of Breaker. Moreover, there are at most two vertices of V_0 whose degree in H is m, and, if there are exactly two such vertices, then they are connected by an edge of Breaker.

For every positive integer i, let $D(i) = \sum_{v \in V_0} d_H(v)$ immediately after Breaker's ith move. Note that $D(i) \geq 0$ for every i and that $D(1) \leq (m-1)n + 2$ (before the game starts the maximum degree of H is at most $m - 1$, and Breaker claims one edge in his first move). For an arbitrary positive integer i, let uv be the edge claimed by Maker in his ith move. At the time it was claimed, we had $d_H(u) = \Delta(H) \geq m - 1$. Assume first that $d_H(v) \geq 2$ was true as well. It follows that $D(i+1) \leq D(i) - (m-1) - (m-1) - 2 - 2 + 2 = D(i) - 2m$ (we subtract $2m + 2$ from $D(i)$ because of u, v and their neighbors, and then add 2 because Breaker claims some edge in his $(i+1)$th move). It follows that there can be at most $\frac{(m-1)n}{2m}$ such moves throughout the first stage. Assume next that $d_H(v) \leq 1$; note that this entails $d_H(v) \leq m - 2$ as $m \geq 3$ by assumption. It follows by Maker's strategy that u is connected by an edge of H to every vertex $x \in V_0$ such that $d_H(x) \geq 2$. Claiming uv decreases $d_H(w)$ by at least 1 for every $w \in V_0 \cap N_H(u)$. It follows by Property ($*$) that after this move of Maker there is at most one vertex $z \in V_0$ such that $d_H(z) \geq m - 1$. It is easy to see that, unless he forfeits the game, Maker can ensure $\Delta(H) \leq m - 2$ in his next move. It follows that Stage I lasts at most $\frac{(m-1)n}{2m} + 2$ moves, as claimed. In particular, we have $|V_0| \geq n/m - 4 > m + 1 \geq \Delta(H) + 1$, and thus Maker can indeed follow Stage I of the proposed strategy without forfeiting the game.

Next, we prove that Maker can follow Stage II of the proposed strategy. Since the first stage lasts at most $\frac{(m-1)n}{2m} + 2$ moves, $|V_0| \geq n/m - 4 \geq n_{m-1}$ holds at the beginning of Stage II. Hence, it follows by the induction hypothesis that Maker can win the positive minimum degree game on $(G \setminus B)[V_0]$ within $\lfloor |V_0|/2 \rfloor + 1$ moves, as claimed. □

Remark 3.5. The requirement $n/m - 4 \geq n_{m-1}$ appearing in the proof of Theorem 3.4 shows that the assertion of this theorem holds even for $m = c \log n / \log \log n$, where $c > 0$ is a sufficiently small constant.

3.3. *A strong positive minimum degree game*

In this subsection, we study the strong version of the *positive minimum degree* game $(E(G), \mathcal{D}_G^1)$. We prove the following result.

Theorem 3.6. *For every positive integer m there exists an integer n_m such that, for every $n \geq n_m$ and for every graph $G = (V, E)$ on n vertices with minimum degree at least $n - m$, Red has a strategy to win the strong positive minimum degree game $(E(G), \mathcal{D}_G^1)$, within at most $\lfloor n/2 \rfloor + 1$ moves.*

Proof. Let $\mathcal{S}_G$ be Maker's strategy for the weak positive minimum degree game $(E(G), \mathcal{D}_G^1)$ whose existence is guaranteed by Theorem 3.4. If n is odd, then Red simply follows $\mathcal{S}_G$. It follows by Theorem 3.4 that Red builds a spanning subgraph of G with positive minimum degree in $\lfloor n/2 \rfloor + 1$ moves. Since there is no such graph with strictly fewer edges, it follows that Red wins the game. Assume then that n is even.

We describe a strategy for Red for the strong positive minimum degree game $(E(G), \mathcal{D}_G^1)$, and then prove that it is a winning strategy. At any point during the game, if Red is unable to follow the proposed strategy, then he forfeits the game. At any point during the game, let $V_0 := \{v \in V : d_R(v) = 0\}$. The strategy is divided into the following five stages.

Stage I: In his first move of this stage, Red claims an arbitrary edge $e_1 = u_1v_1$. Let $f = xy$ denote the edge Blue has claimed in his first move; assume without loss of generality that $x \notin e_1$. Let $A = \{z \in V_0 \setminus \{x\} : xz \notin E\} \cup \{y\}$. For every $i \geq 2$, immediately before his ith move in this stage, Red checks whether $\Delta(B) \geq 2$, in which case he skips to Stage V. Otherwise, Red checks whether $A \cap V_0 = \emptyset$, in which case Stage I is over and Red proceeds to Stage II. Otherwise, let $w \in A \cap V_0$ be an arbitrary vertex. In his ith move in this stage, Red claims a free edge ww' for some $w' \in V_0$.

Stage II: Let $H = (G \setminus B)[V_0 \setminus \{x\}]$, and let $\mathcal{S}_H$ be the winning strategy for Maker in the weak positive minimum degree game, played on $E(H)$, which is described in the proof of Theorem 3.4. Let r denote the total number of moves Red has played in Stage I. For every $r < i \leq 3n/8$, immediately before his ith move in this stage, Red checks whether $\Delta(B) \geq 2$, in which case he skips to Stage V. Otherwise, Red plays his ith move according to the strategy $\mathcal{S}_H$. Once Stage II is over, Red proceeds to Stage III.

Stage III: Let $H = (G \setminus B)[V_0 \setminus \{x\}]$, and let $\mathcal{S}_H$ be the winning strategy for Maker in the weak positive minimum degree game, played on $E(H)$, which is described in the proof of Theorem 3.4. For every $3n/8 < i \leq n/2 - 1$, Red plays his ith move according to the strategy $\mathcal{S}_H$. Once Stage III is over, Red proceeds to Stage IV.

Stage IV: Let $z \in V_0 \setminus \{x\}$. If $xz \in E$ is free, then Red claims it. Otherwise, in his next two moves, Red claims free edges xx' and zz' for some $x', z' \in V$. In either case, the game is over.

Stage V: Let $H = (G \setminus B)[V_0]$, and let $\mathcal{S}_H$ be the winning strategy for Maker in the weak positive minimum degree game, played on $E(H)$, which is described in the proof of Theorem 3.4. In this stage, Red follows $\mathcal{S}_H$ until the end of the game.

We first prove that Red can indeed follow the proposed strategy without forfeiting the game. We consider each stage separately.

Stage I: Since $\delta(G) \geq n-m$, it follows that $|A| \leq m$. Since, moreover, n is sufficiently large with respect to m, we conclude that Red can follow Stage I of the proposed strategy.

Stage II: In Stage I, Maker claims e_1 and then claims at most one additional edge per element of A. It follows that $r \leq |A|+1 \leq m+1$. At the beginning of this stage we have $|V_0 \setminus \{x\}| = n-2r-1 \geq 0.99n$ and $\delta((G \setminus B)[V_0 \setminus \{x\}]) \geq |V_0| - 1 - m - r \geq |V_0| - 2m - 2$. Since n is assumed to be sufficiently large with respect to m, it follows by Theorem 3.4 that the required strategy $\mathcal{S}_H$ exists, and that Red can indeed follow it throughout this stage.

Stage III: At the beginning of this stage we have $|V_0 \setminus \{x\}| \geq n/4 - 1$. Moreover, since Red did not skip to Stage V, it follows that $\delta((G \setminus B)[V_0 \setminus \{x\}]) \geq |V_0| - m - 2$. Since n is assumed to be sufficiently large with respect to m, it follows by Theorem 3.4 that the required strategy $\mathcal{S}_H$ exists, and that Red can indeed follow it throughout this stage.

Stage IV: If the edge xz is still free, then Red can clearly claim it. Otherwise, Red can claim a free edge incident with x and a free edge incident with z, since clearly $\Delta(B) < n/2$.

Stage V: At the beginning of this stage we have $|V_0| \geq n/4$. Moreover, since Red has just skipped to Stage V, it follows that $\delta((G \setminus B)[V_0]) \geq |V_0| - m - 2$. Since n is assumed to be sufficiently large with respect to m, it follows by Theorem 3.4 that the required strategy $\mathcal{S}_H$ exists, and that Red can indeed follow it throughout this stage.

Next, we prove that, if Red follows the proposed strategy, then he wins the game within at most $n/2 + 1$ moves. If Red reaches Stage V of the proposed strategy, then the game lasts at most $n/2 + 1$ moves. Since Red reaches Stage V only after Blue wastes a move, it follows by Theorem 3.4 that Red wins the game in this case. Assume then that Red never reaches Stage V of the proposed strategy. It is clear that, at the end of Stage I, Red's graph is a matching. Moreover, it follows by the proof of Theorem 3.4 that Red's graph is a matching at the end of Stages II and III as well. Moreover, it is clear that $x \in V_0$ holds at this point. Hence, at the beginning of Stage IV, we have $V_0 = \{x, z\}$ for some $z \in V$. Moreover, by Stage I of the proposed strategy we have $xz \in E$. If xz is free, then Red claims it, and thus builds a perfect matching in $n/2$ moves; hence, he wins the game in this case. Otherwise, the game lasts $n/2 + 1$ moves. However, in this case, xz was claimed by Blue, and thus $d_B(x) \geq 2$. We conclude that Red wins the game in this case as well. This concludes the proof of the theorem. □

4. The Maker–Breaker k-vertex-connectivity game

In this section, we prove Theorem 1.1. In our proof we will use the following immediate corollary of Theorem 1.1 from [7].

Corollary 4.1. *Given an integer $n \geq 4$, let $\mathcal{H}_n^+$ be the family of all edge sets of Hamilton cycles with a chord of K_n. If n is sufficiently large, then Maker has a strategy to win $\mathcal{H}_n^+$ in exactly $n+1$ moves.*

Proof of Theorem 1.1. Assume that $k \geq 4$ (recall that, for $k = 2$, the assertion of Theorem 1.1 follows by Theorem 1.1 from [7]; moreover, at the end of the proof we will indicate which small changes have to be made to include the case $k = 3$). We present a strategy for Maker, and then prove it is a winning strategy. At any point during the game, if Maker is unable to follow the proposed strategy, then he forfeits the game. Moreover, if, after claiming kn edges, Maker has not yet built a k-vertex-connected graph, then he forfeits the game (we will in fact prove that Maker can build such a graph much faster; however, the technical upper bound of kn will suffice for the time being). At certain points during the game, Maker will restrict his attention to specific parts of the board. Following some strategy for that part, it might seem like Maker is playing several consecutive moves (as Breaker might decide to respond outside what Maker considers to be the board). Note that this will not cause a problem. Indeed, it is well known (see e.g. [2]) that, if Maker has a winning strategy $\mathcal{S}$ for a weak game $(X, \mathcal{F})$, then he can adjust $\mathcal{S}$ to win $(X, \mathcal{F})$ even if Breaker skips some of his moves. The proposed strategy is divided into the following four stages.

Stage I: Let $V(K_n) = V_1 \cup V_2 \cup \cdots \cup V_{k-1}$ be an arbitrary equipartition of $V(K_n)$ into $k-1$ pairwise disjoint sets, that is, $\||V_i| - |V_j|\| \leq 1$ and $V_i \cap V_j = \emptyset$ for every $1 \leq i \neq j \leq k-1$. For every $1 \leq i \leq k-1$, let $\mathcal{S}_i$ be a winning strategy for Maker in the game $\mathcal{H}_{|V_i|}^+$ played on $E(K_n[V_i])$, whose existence is ensured by Corollary 4.1. In this stage, Maker's goal is to build a Hamilton cycle of $K_n[V_i]$ with a chord for every $1 \leq i \leq k-1$ while limiting the degree of certain vertices in Breaker's graph. If Maker is unable to accomplish both goals within $2n$ moves, then he forfeits the game. For every vertex $v \in V(K_n)$, let $1 \leq i_v \leq k-1$ be the (unique) index such that $v \in V_{i_v}$. Throughout this stage, Maker maintains a set $D \subseteq V(K_n) \times [k-1]$ of *dangerous* pairs. A pair $(v, i) \in V(K_n) \times [k-1]$ is called *dangerous* if $v \notin V_i$, $d_B(v, V_i) \geq 0.9|V_i|$, $d_M(v, V_i) = 0$, and $d_M(v) < k$. Initially, $D = \emptyset$. For every positive integer j, let $e_j = uv$ denote the edge which was claimed by Breaker in his jth move. Maker plays his jth move as follows.

(i) If $e_j \in E(V_i)$ for some $1 \leq i \leq k-1$ and $M[V_i]$ is not yet a Hamilton cycle (of $K_n[V_i]$) with a chord, then Maker responds in this board according to the strategy $\mathcal{S}_i$.
(ii) Otherwise, if $D \neq \emptyset$, let $(z, i) \in D$ be a dangerous pair such that $d_B(z, V_i) = \max\{d_B(w, V_\ell) : (w, \ell) \in D\}$. Maker claims a free edge zw such that $w \in V_i$ and $d_M(w, V_{i_z}) = 0$. Subsequently, Maker updates $D := D \setminus \{(z, V_i), (w, V_{i_z})\}$.
(iii) Otherwise, if there exists $x \in \{u, v\}$ such that $M[V_{i_x}]$ is not yet a Hamilton cycle with a chord, then Maker plays as follows. Assume without loss of generality that $d_B(u, V_{i_v}) \geq d_B(v, V_{i_u})$. If $M[V_{i_u}]$ is not yet a Hamilton cycle with a chord, then Maker follows $\mathcal{S}_{i_u}$ on the board $E(V_{i_u})$; otherwise, he follows $\mathcal{S}_{i_v}$ on $E(V_{i_v})$.
(iv) Otherwise, Maker plays according to $\mathcal{S}_i$ in a board $E(V_i)$ for some $1 \leq i \leq k-1$ such that $M[V_i]$ is not yet a Hamilton cycle with a chord.

As soon as $M[V_i]$ is a Hamilton cycle with a chord for every $1 \leq i \leq k-1$ and $D = \emptyset$, this stage is over, and Maker proceeds to Stage II.

Stage II: Let C be the set of endpoints of the chords of $\bigcup_{i=1}^{k-1} M[V_i]$. At any point during this stage, let $Y_C = \{v \in C : d_M(v) < k\}$, let $Y_D = \{v \in V(K_n) : d_M(v) < k$ and $d_B(v) \geq k^{10}\}$, and let $Y = Y_C \cup Y_D$. For as long as $Y \neq \emptyset$, Maker picks an arbitrary vertex $v \in Y$ and plays as follows. Let $t = d_M(v)$, and let $\{i_1, \ldots, i_{k-t}\} \subseteq [k-1] \setminus \{i_v\}$ be $k-t$ distinct indices such that $d_M(v, V_{i_j}) = 0$ for every $1 \leq j \leq k-t$. In his next $k-t$ moves, Maker claims $k-t$ free edges $\{vv_{i_j} : 1 \leq j \leq k-t\}$ such that $v_{i_j} \in V_{i_j}$, $d_M(v_{i_j}) < k$ and $d_M(v_{i_j}, V_{i_v}) = 0$ for every $1 \leq j \leq k-t$.

As soon as $Y = \emptyset$, this stage is over, and Maker proceeds to Stage III.

Stage III: For every $1 \le i \ne j \le k-1$, let $A_{ij} \subseteq V_i$ denote the set of vertices $v \in V_i$ such that $d_M(v) < k$ and $d_M(v, V_j) = 0$. Moreover, for every $1 \le i \ne j \le k-1$, let $B_{ij} \subseteq A_{ij}$ be sets which satisfy all of the following properties.

(P1) $B_{ij} \cap B_{i\ell} = \emptyset$ for every $1 \le i \le k-1$ and for every $1 \le j \ne \ell \le k-1$.
(P2) $10k^{-5}|A_{ij}| \le |B_{ij}| \le 11k^{-5}|A_{ij}|$ for every $1 \le i \ne j \le k-1$.
(P3) $dist_{M[V_i]}(u, v) \ge 2$ for every $1 \le i \le k-1$ and for every two distinct vertices $u, v \in \bigcup_{j \in [k-1] \setminus \{i\}} B_{ij}$.

For every $1 \le i < j \le k-1$, let $G_{ij} = (A_{ij} \cup A_{ji}, E_{K_n \setminus B}(A_{ij}, A_{ji}))$, and let $\mathcal{S}_{ij}$ be the winning strategy for Maker in the game $G_{ij}(A_{ij}, B_{ij}; A_{ji}, B_{ji}; 2k^{10})$, which is described in the proof of Lemma 3.1.

At any point during this stage, for every $1 \le i < j \le k-1$, Maker maintains a matching M_{ij} of the board $E(G_{ij})$ and a set $D \subseteq V(K_n)$ of *dangerous* vertices. A vertex $v \in V(K_n)$ is called dangerous if $v \in B_{ij}$ for some $1 \le i \ne j \le k-1$ (without loss of generality assume that $i < j$) and, moreover, v satisfies all of the following properties.

(1) v is not matched in M_{ij}.
(2) M_{ij} covers $(A_{ij} \setminus B_{ij}) \cup (A_{ji} \setminus B_{ji})$.
(3) $d_B(v) \ge k^{10}$.

Initially, $D = M_{ij} = \emptyset$ for every $1 \le i < j \le k-1$.

Let r denote the number of moves Maker has played throughout Stages I and II. For every $s > r$, let e_s denote the edge that was claimed by Breaker in his sth move. Maker plays his sth move as follows.

(i) If $e_s \in E(G_{ij})$ for some $1 \le i < j \le k-1$ and M_{ij} does not yet cover $\left(A_{ij} \setminus B_{ij}\right) \cup \left(A_{ji} \setminus B_{ji}\right)$, then Maker responds in the board $E(G_{ij})$ according to the strategy $\mathcal{S}_{ij}$.
(ii) Otherwise, if $D \ne \emptyset$, then Maker claims a free edge uv between two sets B_{ij} and B_{ji} such that the following properties hold.
   (a) $u \in D$.
   (b) $d_B(u) = \max\{d_B(w) : w \in D\}$.
   (c) M_{ij} covers $(A_{ij} \setminus B_{ij}) \cup (A_{ji} \setminus B_{ji})$.
   (d) v is not covered by M_{ij}.
   Maker updates $D := D \setminus \{u, v\}$.
(iii) Otherwise, Maker picks arbitrarily $1 \le i < j \le k-1$ such that M_{ij} does not yet cover $(A_{ij} \setminus B_{ij}) \cup (A_{ji} \setminus B_{ji})$ and plays in the board $E(G_{ij})$ according to the strategy $\mathcal{S}_{ij}$.

As soon as M_{ij} covers $\left(A_{ij} \setminus B_{ij}\right) \cup \left(A_{ji} \setminus B_{ji}\right)$ for every $1 \le i < j \le k-1$ and $D = \emptyset$, this stage is over, and Maker proceeds to Stage IV.

Stage IV: Let $U = \{v \in V(K_n) : d_M(v) = k-1\}$ and let $H := (K_n \setminus (B \cup M))[U]$. Let $\mathcal{S}_H$ be a strategy for Maker to win the positive minimum degree game $(E(H), \mathcal{D}^1_H)$ within $\lfloor |U|/2 \rfloor + 1$ moves. In this stage, Maker follows $\mathcal{S}_H$ until $\delta(M) \ge k$ first occurs; at this point, the game is over.

It is evident that, if Maker can follow the proposed strategy without forfeiting the game, then, by the end of the game, he builds a graph $M \in \mathcal{G}_k$, which is k-vertex-connected by Proposition 2.1. Indeed, Property (i) in the description of $\mathcal{G}_k$ is satisfied since n is sufficiently large with respect to k. Property (ii) is satisfied since $\delta(M) \ge k-1$ and $U \subseteq \bigcup_{1 \le i \ne j \le k-1} B_{ij}$ hold at the beginning of Stage IV. Property (iii) is satisfied by Stage I. Property (iv) is satisfied since all the sets B_{ij} are small. Property (v) is satisfied by (P1) in the definition of the sets B_{ij} and Property (vi) is satisfied by (P3) in the definition of the sets B_{ij}.

It thus suffices to prove that Maker can indeed follow the proposed strategy without forfeiting the game and that, by doing so, he builds an element of $\mathcal{G}_k$ within $\lfloor kn/2 \rfloor + 1$ moves.

We now prove that Maker can indeed follow the proposed strategy, including the time constraints it sets, without forfeiting the game. We consider each stage separately.

Stage I: Since $|V_i| \ge \lfloor n/(k-1) \rfloor$ for every $1 \le i \le k-1$, and since n is sufficiently large with respect to k, it follows by Corollary 4.1 that Maker can follow Part (i) of the proposed strategy for this stage.

Recall that, by definition, this stage lasts at most $2n$ moves, and that $d_B(v, V_i) \ge 0.9|V_i| \ge 0.9n/k$ holds for every dangerous pair $(v, i) \in D$. Therefore, throughout Stage I, Breaker can create at most $4n/\left(\frac{0.9n}{k}\right) \le 5k$ such pairs. We claim that, at any point during Stage I, $d_B(v, V_i) \le 0.95|V_i|$ holds for every vertex $v \in V(K_n)$ and every $i \in [k-1] \setminus \{i_v\}$. This is immediate by the definition of D for every pair $(v, i) \in (V(K_n) \times [k-1]) \setminus D$. Consider a point during this stage where $D \neq \emptyset$ (if this never happens, then there is nothing left to prove). If Breaker plays in $\bigcup_{i=1}^{k-1} E(V_i)$, then he does not increase $d_B(v, V_i)$ for any pair $(v, i) \in D$. Otherwise, Maker follows Part (ii) of the proposed strategy for this stage and thus decreases the size of D. It follows that, throughout Stage I, Maker follows Part (ii) of the proposed strategy at most $5k$ times. Since n is sufficiently large with respect to k, it follows that, throughout Stage I, $d_B(v, V_i) \leq 0.9|V_i| + 5k \leq 0.95|V_i|$ holds for every $v \in V(K_n)$ and every $i \in [k-1] \setminus \{i_v\}$, as claimed. Since Maker follows Part (ii) of the proposed strategy at most $5k$ times, and since he only claims edges of $\bigcup_{i=1}^{k-1} E(V_i)$ when following Part (i), (iii), or (iv) of the strategy, it follows that, throughout Stage I, $|\{u \in V_i : d_M(u, V_j) = 0\}| \geq 0.99|V_i|$ holds for every $1 \leq i \neq j \leq k-1$. Hence, Maker can follow Part (ii) of the proposed strategy for this stage without forfeiting the game.

Finally, it readily follows from Corollary 4.1 that Maker can follow Parts (iii) and (iv) of the proposed strategy for this stage.

It thus suffices to prove that Maker can achieve his goals for this stage within at most $2n$ moves. This readily follows from the following three simple observations.

(a) According to Corollary 4.1, for every $1 \leq i \leq k-1$, Maker can build a Hamilton cycle of $K_n[V_i]$ with a chord in $|V_i| + 1$ moves.
(b) Whenever Maker follows Part (i), (iii), or (iv) of the proposed strategy for this stage, he plays according to $\mathcal{S}_i$ for some $1 \leq i \leq k-1$.
(c) As previously noted, Maker follows Part (ii) of the proposed strategy at most $5k$ times.

It follows that Stage I lasts at most $\sum_{i=1}^{k-1} (|V_i| + 1) + 5k = n + (k-1) + 5k < 2n$ moves.

We conclude that Maker can follow the proposed strategy for this stage, including the time limits it sets, without forfeiting the game.

Stage II: Since the entire game lasts at most kn moves, it follows that $|\{u \in V(K_n) : d_B(u) \geq k^{10}\}| \leq 2kn/k^{10}$ holds at any point during the game. Hence, $|Y| \leq 2(k-1) + 2n/k^9 \leq 3n/k^9$ holds at any point during this stage. Since $D = \emptyset$ at the end of Stage I, and since Maker spends at most k moves on every vertex of Y, it follows that, at any point during this stage, $d_B(v, V_i) \leq 0.9|V_i| + 3n/k^8 \leq 0.95|V_i|$ holds for every vertex $v \in Y$ and for every $i \in [k-1] \setminus \{i_v\}$. Since, as noted above, $|\{u \in V_i : d_M(u, V_j) = 0\}| \geq 0.99|V_i|$ holds for every $1 \leq i \neq j \leq k-1$ at the end of Stage I, it follows that $|\{u \in V_i : d_M(u, V_j) = 0\}| \geq 0.98|V_i|$ holds for every $1 \leq i \neq j \leq k-1$ throughout Stage II. We conclude that Maker can follow the proposed strategy for this stage without forfeiting the game.

Stage III: For every $1 \leq i \leq k-1$, let $A_i := \{u \in V_i : d_M(u) = 2\}$. Since Maker follows Part (ii) of Stage I at most $5k$ times, and since Stage II lasts at most $3n/k^8$ moves, we conclude that $|A_i| \geq \lfloor n/(k-1) \rfloor - 2 - 5k - 3n/k^8 \geq 0.99n/(k-1)$ holds for every such i. Moreover, since $2 + 5k + 3n/k^8 \leq n/k^7$, it follows that $||A_{ij}| - |A_{ji}|| \leq n/k^7$ holds for every $1 \leq i < j \leq k-1$.

For every $1 \leq i \leq k-1$, let $B_i \subseteq A_i$ be a set which satisfies $|B_i| \geq \lfloor |A_i|/2 \rfloor \geq |A_i|/3$ and $dist_{M[V_i]}(u, v) \geq 2$ for every $u, v \in B_i$ (one example of such a set is obtained by enumerating the elements of A_i according to their order of appearance on the Hamilton cycle of $K_n[V_i]$ and taking either all even indexed vertices or all odd indexed vertices). Let $B_i = B_i^{(1)} \cup \cdots \cup B_i^{(i-1)} \cup B_i^{(i+1)} \cup \cdots \cup B_i^{(k-1)}$ be an equipartition of B_i. For every $1 \leq i < j \leq k-1$, let $B_{ij} \subseteq B_i^{(j)}$ and $B_{ji} \subseteq B_j^{(i)}$ be chosen such that Property (P2) in the description of the proposed strategy for this stage holds. Note that Properties (P1) and (P3) hold as well by the construction of the sets B_i and the $B_i^{(j)}$.

Since, as noted above, $||A_{ij}| - |A_{ji}|| \leq n/k^7$ holds for every $1 \leq i < j \leq k-1$, since $d_B(u) < k^{10}$ holds for every $u \in A_i$ by Stage II of the proposed strategy, and since n is sufficiently large with respect to k, it follows by Lemma 3.1 (with $\varepsilon = k^{-5}$) that Maker can follow Parts (i) and (iii) of the proposed strategy for this stage.

Moreover, since $d_B(v) \geq k^{10}$ holds for every dangerous vertex, and since the entire game lasts at most kn moves, it follows that Breaker can create at most $2kn/k^{10} \leq n/k^8$ such vertices. Since Maker spends exactly one move to treat a dangerous vertex, and since $|B_{ij}| \geq 10k^{-5}|A_{ij}| \geq 10k^{-5}|A_i| \geq n/k^6$ holds by construction for every $1 \leq i \neq j \leq k-1$, it follows that Maker can indeed follow Part (ii) of the proposed strategy for this stage.

Stage IV: Since, as noted in the previous paragraph, Breaker can create at most $2kn/k^{10}$ dangerous vertices throughout the game, it follows that Maker plays according to the proposed strategy for Part (ii) of Stage III at most $2n/k^9$ times. It follows by Lemma 3.1 and by Property (P2) that

$$|U| \geq \sum_{1\leq i\neq j\leq k-1} |B_{ij}|/2 - 4n/k^9 \geq \binom{k-1}{2}\frac{10n}{2k^6} - 4n/k^9 \geq n/k^4.$$

Since n is sufficiently large with respect to k, it thus follows by Theorem 3.4 that Maker can follow the strategy $\mathcal{S}_H$ throughout this stage without forfeiting the game.

It remains to prove that, by following the proposed strategy, Maker wins the game within $\lfloor kn/2\rfloor + 1$ moves. It follows by Theorem 3.4 that Stage IV lasts at most $\lfloor |U|/2\rfloor + 1$ moves. It thus suffices to prove that $\Delta(M) \leq k$ holds throughout Stages I–III. This follows quite easily from the description of Maker's strategy. There is one exception though. It is theoretically possible that at some point during the game there will be some $1 \leq i \leq k-1$ and a vertex $u \in V_i$ such that $d_M(u, V_j) > 0$ will hold for every $j \in [k-1] \setminus \{i\}$, but at that point it will not yet be clear whether u will be an endpoint of the chord of the Hamilton cycle in $M[V_i]$. If u will indeed become an endpoint of the chord, then its degree in Maker's graph will be $k+1$. In order to overcome this problem, we include Part (iii) of the proposed strategy for Stage I. We claim that this situation cannot occur, that is, that for every $1 \leq i \leq k-1$ and every $u \in V_i$, if $d_M(u, V_j) > 0$ holds for every $j \in [k-1] \setminus \{i\}$, then $M[V_i]$ is a Hamilton cycle with a chord. Assume for the sake of contradiction that such i and u exist. Recall that $k \geq 4$ by assumption. It follows that there are indices $1 \leq j_1 < j_2 \leq k-1$ such that $i \notin \{j_1, j_2\}$, $d_M(u, V_{j_1}) > 0$ and $d_M(u, V_{j_2}) > 0$. Since Maker follows the proposed strategy, it follows by the definition of dangerous pairs that $d_B(u, V_{j_1}) \geq 0.9|V_{j_1}|$ and $d_B(u, V_{j_2}) \geq 0.9|V_{j_2}|$. Recall that Stage I lasts at most $2n$ moves. Therefore, for $t \in \{1, 2\}$, there are at most $\frac{4n}{0.1n/k} = 40k$ vertices of $N_B(u, V_{j_t})$ of degree at least $0.1n/k$ in Breaker's graph. It follows by Part (iii) of the proposed strategy for Stage I, that, for at least $0.9|V_{j_1}| + 0.9|V_{j_2}| - 0.2n/k - 80k > 1.5n/k$ of the times Breaker claims an edge uv such that $v \in V_{j_1} \cup V_{j_2}$, Maker plays in $E(V_i)$. This contradicts the description of the proposed strategy. For $k = 3$, we have no choice but to ensure that, if a vertex u satisfies $d_M(u, V_i)$ for $i \neq i_u$, then it will not become an endpoint of the chord of $M[V_{i_u}]$. In order to ensure this, one has to slightly alter Maker's strategy for the game $\mathcal{H}^+_{|V_{i_u}|}$. This can be done by adjusting the strategy given in the proof of Theorem 1.1 in [7] or the strategy given in the proof of Theorem 1.1 in [6] (the latter is easier). Note that this solution works for every $k \geq 3$. However, where possible, we preferred a solution which uses Maker's strategy for the Hamilton cycle with a chord game as a black box.

This concludes the proof of the theorem. □

5. The strong *k*-vertex-connectivity game

Proof of Theorem 1.3. Let $k \geq 3$ be an integer, and assume first that kn is odd. Red simply follows Maker's strategy for the weak k-vertex-connectivity game $(E(K_n), \mathcal{C}_n^k)$, whose existence is guaranteed by Theorem 1.1. It follows by Theorem 1.1 that he builds a k-vertex-connected graph in $\lfloor kn/2\rfloor + 1$ moves. Since, for odd kn, there is no graph G on n vertices such that $\delta(G) \geq k$ and $e(G) \leq \lfloor kn/2\rfloor$, it follows that Red wins the strong k-vertex-connectivity game $(E(K_n), \mathcal{C}_n^k)$.

Assume then that kn is even. First, we present a strategy for Red, and then we prove that it is a winning strategy. At any point during the game, if Red is unable to follow the proposed strategy, then he forfeits the game. At certain points during the game, Red will restrict his attention to specific parts of the board. Following some strategy for that part, it might seem like Red is playing several consecutive moves (as Blue might decide to respond outside what Red considers to be the board). Note that this will not cause a problem. Indeed, it is well known (see e.g. [2]) that, if Red has a winning strategy $\mathcal{S}$ for a strong game $(X, \mathcal{F})$, then he can adjust $\mathcal{S}$ to win $(X, \mathcal{F})$ even if Blue skips some of his moves. The proposed strategy is divided into the following two stages.

Stage I: Let $\mathcal{S}_M$ be the winning strategy for Maker in the weak game $(E(K_n), \mathcal{C}_n^k)$, which is described in the proof of Theorem 1.1. In this stage, Red follows Stages I–III of the strategy $\mathcal{S}_M$. As soon as Red first reaches Stage IV of $\mathcal{S}_M$, this stage is over, and Red proceeds to Stage II.

Stage II: Let $U_0 := \{v \in V(K_n) : d_R(v) = k-1\}$, and let $G = (K_n \setminus (B \cup M))[U_0]$. Let $\mathcal{S}_G$ be the winning strategy for Red in the strong positive minimum degree game $(E(G), \mathcal{D}_G^1)$, which is described in the proof of Theorem 3.6. We distinguish between the following three cases.

(1) If $\Delta(B) > k$, then Red continues playing according to the strategy $\mathcal{S}_M$ until the end of the game. That is, he follows Stage IV of $\mathcal{S}_M$ until his graph first becomes k-vertex-connected.
(2) Otherwise, if $d_B(v) \leq k-1$ for every $v \in U_0$, then Red plays the strong positive minimum degree game $(E(G), \mathcal{D}_G^1)$ according to the strategy $\mathcal{S}_G$ until his graph first becomes k-vertex-connected.
(3) Otherwise, let $x \in U_0$ be a vertex such that $d_B(x) = k$. This case is further divided into the following five substages.
   (i) Let r_1 denote the total number of moves Red has played thus far. For every $i > r_1$, immediately before his ith move, Red checks whether $\Delta(B) > k$, in which case he skips to Substage (v). Otherwise, Red checks whether $N_B(x) \cap U_0 = \emptyset$, in which case Substage (i) is over, and Red proceeds to Substage (ii). Otherwise, let $w \in N_B(x) \cap U_0$ be an arbitrary vertex. In his ith move, Red claims a free edge ww' for some $w' \in U_0$.
   (ii) Let r_2 denote the number of moves Red has played in Substage (i), and let $r = r_1 + r_2$. Let $U_0' := \{v \in V(K_n) : d_R(v) = k-1\}$, let $H = (K_n \setminus (B \cup M))[U_0' \setminus \{x\}]$, and let $\mathcal{S}_H$ be the winning strategy for Red in the strong positive minimum degree game $(E(H), \mathcal{D}_H^1)$, which is described in the proof of Theorem 3.6. For every $r < i \leq kn/2 - |U_0|/3$, immediately before his ith move, Red checks whether $\Delta(B) > k$, in which case he skips to Substage (v). Otherwise, Red plays his ith move according to the strategy $\mathcal{S}_H$. As soon as this substage is over, Red proceeds to Substage (iii).
   (iii) For every $kn/2 - |U_0|/3 < i \leq kn/2 - 1$, Red plays his ith move according to the strategy $\mathcal{S}_H$. When this substage is over, Red proceeds to Substage (iv).
   (iv) Let $z \in U_0' \setminus \{x\}$ be a vertex of degree $k-1$ in Red's graph. If the edge $xz \in E(K_n)$ is free, then Red claims it. Otherwise, in his next two moves, Red claims free edges xx' and zz' for some $x', z' \in V(K_n)$. In both cases, the game is over.
   (v) Let $U := \{v \in V(K_n) : d_R(v) = k-1\}$, and let $G' = (K_n \setminus (B \cup M))[U]$. Let $\mathcal{S}_{G'}$ be the winning strategy for Red in the strong positive minimum degree game $(E(G'), \mathcal{D}_{G'}^1)$, which is described in the proof of Theorem 3.6. In this substage, Red follows $\mathcal{S}_{G'}$ until the end, that is, until his graph first becomes k-vertex-connected.

It is evident that, if Red can follow the proposed strategy without forfeiting the game, then, by the end of the game, he builds a graph $R \in \mathcal{G}_k$, which is k-vertex-connected by Proposition 2.1. It thus suffices to prove that Red can indeed follow the proposed strategy without forfeiting the game, that he builds an element of $\mathcal{G}_k$ within $\lfloor kn/2 \rfloor + 1$ moves, and that he does so before $\delta(B) \geq k$ first occurs.

Our first goal is to prove that Red can indeed follow the proposed strategy without forfeiting the game. We consider each stage separately.

Stage I: Since n is sufficiently large with respect to k, it follows by Theorem 1.1 that Red can follow Stage I of the proposed strategy.

Stage II: We consider each of the three cases separately.

(1) Since Red has played all of his moves in Stage I according to the strategy $\mathcal{S}_M$, it follows by the proof of Theorem 1.1 that he can continue doing so until the end of the game.
(2) Since Red has played all of his moves in Stage I according to the strategy $\mathcal{S}_M$, it follows by the proof of Theorem 1.1 that $|U_0| = \Omega(n)$ holds at the beginning of Stage II. Since we are not in Case (1), it follows by Red's strategy that $\delta(G) \geq |U_0| - 2k$. Since, moreover, n is sufficiently large with respect to k, it follows by Theorem 3.6 that Red can indeed follow the proposed strategy for this case without forfeiting the game.
(3) As previously noted, $|U_0| = \Omega(n)$ and $\delta(G) \geq |U_0| - 2k$ hold at the beginning of Stage II. Since Red skips to Substage (v) as soon as $\Delta(B) > k$ first holds, it follows that Red can follow the proposed strategy for Substage (i), and that this substage lasts $r_2 \leq d_B(x) \leq k$ moves. It thus follows that $|U_0'| = \Omega(n)$ and that $\delta(H) \geq |U_0'| - 1 - 2k$ holds throughout Substage (ii). Since, moreover, n is sufficiently large with respect to k, it follows by Theorem 3.6 that Red can follow Substage (ii) of the proposed strategy for this case. Since $\Delta(B) \leq k$ holds at the beginning of Substage (iii)

(otherwise Red would have skipped to Substage (v)), it follows by an analogous argument that Red can follow Substage (iii) of the proposed strategy for this case as well. It follows by Substages (ii) and (iii) of the proposed strategy that, at the beginning of Substage (iv), there are exactly two vertices of degree $k-1$ in Red's graph, one of which is x. Denote the other one by z. Since $\Delta(B) \leq k$ holds at the beginning of Substage (iii), and since Substage (iii) clearly lasts at most $|U_0|/3$ moves, it follows that $d_B(x) \leq k + |U_0|/3 < n/2$ and $d_B(z) \leq k + |U_0|/3 < n/2$ hold at the beginning of Substage (iv). Hence, Red can follow Substage (iv) of the proposed strategy for this case. Finally, since $\Delta(B) \leq k + 1$ and $|U| = \Omega(n)$ clearly hold at the beginning of Substage (v), and since n is sufficiently large with respect to k, it follows by Theorem 3.6 that Red can follow Substage (v) of the proposed strategy for this case.

It is evident from the description of the proposed strategy that the game lasts at most $kn/2 + 1$ moves. Hence, in order to complete the proof of the theorem, it suffices to show that, if the game lasts exactly $kn/2 + 1$ moves, then $\Delta(B) > k$. This clearly holds if the game ends in Case (1) or in Substage (v) of Case (3). If the game ends in Case (2), then this follows by Theorem 3.6. Finally, if the game lasts exactly $kn/2 + 1$ moves and ends in Substage (iv) of Case (3), then $d_B(x) \geq k + 1$ must hold by the proposed strategy for Substages (i) and (iv) of this case.

This concludes the proof of the theorem. $\square$

6. Concluding remarks and open problems

A more natural fastest possible strategy for the minimum-degree-k game. As noted in Corollary 1.2 (respectively, Corollary 1.4), Maker (respectively, Red) can win the weak (respectively, strong) minimum-degree-k game $(E(K_n), \mathcal{D}_n^k)$ within $\lfloor kn/2 \rfloor + 1$ moves by following his strategy for the weak (respectively, strong) k-vertex-connectivity game $(E(K_n), \mathcal{C}_n^k)$. While useful, this is not a very natural way to play this game. We have found a much more natural strategy for Maker (respectively, Red) to win the weak (respectively, strong) game $(E(K_n), \mathcal{D}_n^k)$ within $\lfloor kn/2 \rfloor + 1$ moves. It consists of two main stages. In the first stage, Maker (respectively, Red) builds a graph with minimum degree $k-1$ and maximum degree k. This is done almost arbitrarily, except that Maker (respectively, Red) ensures that, if a vertex has degree $k-1$ in his graph, then its degree in Breaker's (respectively, Blue's) graph will not be too large. In the second stage, he plays the weak (respectively, strong) positive minimum degree game $(E(K_n), \mathcal{D}_n^1)$ on the graph induced by the vertices of degree $k-1$ in his graph. We omit the details.

Explicit winning strategies for other strong games. Following the observation made in [3] that fast winning strategies for Maker in a weak game have the potential of being upgraded to winning strategies for Red in the corresponding strong game, we have devised a winning strategy for Red in the strong k-vertex-connectivity game. It is plausible that one could devise a winning strategy for other strong games, where a fast strategy is known for the corresponding weak game. One natural candidate is the *specific spanning tree* game. This game is played on the edge set of K_n for some sufficiently large integer n. Given a tree T on n vertices, the family of winning sets $\mathcal{T}_n$ consists of all copies of T in K_n. It was proved in [4] that Maker has a strategy to win the weak game $(E(K_n), \mathcal{T}_n)$ within $n + o(n)$ moves provided that $\Delta(T)$ is not too large.

On the other hand, there are weak games for which Maker has a winning strategy and yet Breaker can avoid losing them quickly. Consider for example the *Clique* game $RG(n, q)$. The board of this game is the edge set of K_n, and the family of winning sets consists of all copies of K_q in K_n. It is easy to see that for every positive integer q there exists an integer n_0 such that Maker (respectively, Red) has a strategy to win the weak (respectively, strong) game $RG(n, q)$ for every $n \geq n_0$. However, it was proved in [1] that Breaker can avoid losing this game during the first $2^{q/2}$ moves. The current best upper bound on the number of moves needed for Maker in order to win $RG(n, q)$ is $2^{2q/3} \cdot f(q)$, where $f(q)$ is some polynomial in q (see [5]). Note that this upper bound does not depend on the size of the board; in particular, it holds for an infinite board as well. Given that an exponential lower bound on the number of moves is known, it would be very interesting to find an explicit winning strategy for Red in the strong game $RG(n, q)$ for every positive integer q and sufficiently large n. Moreover, it would be interesting to determine whether Red can win this game on an infinite board.

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